

# A NEW PROOF OF THE FUNDAMENTAL TWO-TERM TRANSFORMATION FOR THE SERIES ${}_3F_2(1)$ DUE TO THOMAE

ARJUN K. RATHIE

ABSTRACT. The aim of this short note is to provide a very simple proof for obtaining the fundamental two-term transformation for the series  ${}_3F_2(1)$  due to Thomae.

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## 1. INTRODUCTION AND RESULTS REQUIRED

We start with the following very useful and interesting Thomae[7] and Kummer[1] two-term transformations respectively as

$${}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] = \frac{\Gamma(d)\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(a)\Gamma(d+e-a-b)\Gamma(d+e-a-c)} \cdot {}_3F_2 \left[ \begin{matrix} d-a, e-a, d+e-a-b-c \\ d+e-a-b, d+e-a-c \end{matrix} ; 1 \right] \quad (1)$$

provided  $\Re(a) > 0$  and  $\Re(d+e-a-b-c) > 0$  and

$${}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-c)\Gamma(d+e-a-b)} \cdot {}_3F_2 \left[ \begin{matrix} d-a, d-b, c \\ d, d+e-a-b \end{matrix} ; 1 \right] \quad (2)$$

provided  $\Re(e-c) > 0$  and  $\Re(d+e-a-b-c) > 0$ .

As given in Bailey's tract[2], the Thomae transformation (1) can be established with the help of following classical Gauss's summation theorem[2, p.2, Eq. (1) or 5]

$${}_2F_1 \left[ \begin{matrix} a, b \\ c \end{matrix} ; 1 \right] = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} \quad (3)$$

provided  $\Re(c-a-b) > 0$ .

For a derivation of Kummer's transformation (2) we refer the standard text of Andrews, et al[1, eq. 3.3.5, p.142].

In 1999, by considering the double series and summing up in two ways by employing Gauss's summation theorem (3) and following Saalshütz summation theorem [5]

$${}_3F_2 \left[ \begin{matrix} -n, a, b \\ c, 1+a+b-c-n \end{matrix} ; 1 \right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} \quad (4)$$

Exton[4] re-derived the Kummer's transformation (2).

In 2004, Rathie, et al.[6] have given a very short proof of Kummer's transformation (2).

The aim of this short note is to provide a very simple proof for obtaining the fundamental two-term transformation (1) for the series  ${}_3F_2(1)$  due to Thomae.

For this, we require the following result known as Euler's second transformation[5] :

$${}_2F_1 \left[ \begin{matrix} a, b \\ d \end{matrix} ; x \right] = (1-x)^{d-a-b} {}_2F_1 \left[ \begin{matrix} d-a, d-b \\ d \end{matrix} ; x \right] \quad (5)$$

## 2. DERIVATION OF THOMAE TRANSFORMATION (1)

In order to derive Thomae transformation (1), we start with the following known integral involving hypergeometric function[3, Equ.(5), p. 399] :

$$\int_0^1 x^{c-1} (1-x)^{e-c-1} {}_2F_1 \left[ \begin{matrix} a, b \\ d \end{matrix} ; x \right] dx = \frac{\Gamma(c)\Gamma(e-c)}{\Gamma(e)} {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] \quad (6)$$

provided  $\Re(c) > 0$ ,  $\Re(e-c) > 0$  and  $\Re(d+e-a-b-c) > 0$

In order to prove (6), express  ${}_2F_1$  as a series, change the order of integration and summation, which is justified due to uniform convergence of the series, evaluate the integral with the help of the following beta integral :

$$\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (7)$$

provided  $\Re(\alpha) > 0$  and  $\Re(\beta) > 0$ .

and then summing up the series, we easily arrive at the right-hand side of (6).

Now, in order to establish Thomae transformation (1), consider the result (6) in the form :

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] \\ &= \frac{\Gamma(e)}{\Gamma(c)\Gamma(e-c)} \int_0^1 x^{c-1} (1-x)^{e-c-1} {}_2F_1 \left[ \begin{matrix} a, b \\ d \end{matrix} ; x \right] dx \end{aligned} \quad (8)$$

Applying the result (5) to the  ${}_2F_1$  on the right-hand side, we have

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] \\ &= \frac{\Gamma(e)}{\Gamma(c)\Gamma(e-c)} \int_0^1 x^{c-1} (1-x)^{d+e-a-b-c-1} {}_2F_1 \left[ \begin{matrix} d-a, d-b \\ d \end{matrix} ; x \right] dx \end{aligned} \quad (9)$$

Now, expressing  ${}_2F_1$  as a series, change the order of integration and summation, which is easily seen to be justified due to uniform convergence of the series, we have

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] \\ &= \frac{\Gamma(e)}{\Gamma(c) \Gamma(e-c)} \sum_{n=0}^{\infty} \frac{(d-a)_n (d-b)_n}{(d)_n n!} \cdot \int_0^1 x^{c+n-1} (1-x)^{d+e-a-b-c-1} dx \end{aligned} \quad (10)$$

Evaluating the integral with the help of (7), we have

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] \\ &= \frac{\Gamma(e) \Gamma(d+e-a-b-c)}{\Gamma(e-c) \Gamma(d+e-a-b)} \sum_{n=0}^{\infty} \frac{(d-a)_n (d-b)_n (c)_n}{(d)_n (d+e-a-b)_n n!}. \end{aligned} \quad (11)$$

Summing up the series, we have

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] \\ &= \frac{\Gamma(e) \Gamma(d+e-a-b-c)}{\Gamma(e-c) \Gamma(d+e-a-b)} {}_3F_2 \left[ \begin{matrix} d-a, d-b, c \\ d, d+e-a-b \end{matrix} ; 1 \right] \end{aligned} \quad (12)$$

Now, writing (12) in the form

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] \\ &= \frac{\Gamma(e) \Gamma(d+e-a-b-c)}{\Gamma(e-c) \Gamma(d+e-a-b)} {}_3F_2 \left[ \begin{matrix} d-b, c, d-a \\ d+e-a-b, d \end{matrix} ; 1 \right] \end{aligned} \quad (13)$$

Now using (6) to the  ${}_3F_2$  on the right-hand side of (13), we have

$$\begin{aligned} {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] &= \frac{\Gamma(d) \Gamma(e) \Gamma(d+e-a-b-c)}{\Gamma(a) \Gamma(d-a) \Gamma(e-c) \Gamma(d+e-a-b)} \\ &\quad \cdot \int_0^1 x^{d-a-1} (1-x)^{a-1} {}_2F_1 \left[ \begin{matrix} d-b, c \\ d+e-a-b \end{matrix} ; x \right] dx \end{aligned} \quad (14)$$

Using Euler's second transformation, we have

$$\begin{aligned} {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] &= \frac{\Gamma(d) \Gamma(e) \Gamma(d+e-a-b-c)}{\Gamma(a) \Gamma(d-a) \Gamma(e-c) \Gamma(d+e-a-b)} \\ &\quad \cdot \int_0^1 x^{d-a-1} (1-x)^{e-c-1} {}_2F_1 \left[ \begin{matrix} e-a, d+e-a-b-c \\ d+e-a-b \end{matrix} ; x \right] dx \end{aligned} \quad (15)$$

Finally, expressing  ${}_2F_1$  as a series, change the order of integration and summation and after some algebra, evaluating the beta integral, we have

$${}_3F_2 \left[ \begin{matrix} a, b, c \\ d, e \end{matrix} ; 1 \right] = \frac{\Gamma(d) \Gamma(e) \Gamma(d+e-a-b-c)}{\Gamma(a) \Gamma(d+e-a-b) \Gamma(d+e-a-c)} \cdot \sum_{n=0}^{\infty} \frac{(d-a)_n (e-a)_n (d+e-a-b-c)_n}{(d+e-a-b)_n (d+e-a-c)_n n!} \quad (16)$$

Finally, summing up the series, we get the required Thomae transformation formula (1). This completes the proof of the Thomae transformation formula.

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ARJUN K. RATHIE, DEPARTMENT OF MATHEMATICS, CENTRAL UNIVERSITY OF KERALA, KASARGAD 671123, KERALA, INDIA

*E-mail address:* [akrathie@cukerala.ac.in](mailto:akrathie@cukerala.ac.in)